Exercises for Introduction to Quantum Computing (physics7514) Wise 2025

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1 Staggering

Note that what is referred to here as staggering or staggered fermions is **not** what is generally known in the literature as staggered fermions. The transformation procedure shown in this sheet is more akin to what one has to do for Wilson fermions.

2 Mapping 1+1D Quantum Electrodynamics to Qubits

The continuum Lagrangian of 1+1D Quantum Electrodynamics (QED) reads

$$L = \int \mathrm{d}x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left(i\gamma^\mu \partial_\mu - g\gamma^\mu A_\mu - m \right) \psi \right], \tag{1}$$

where m is the fermion mass, g is the gauge coupling, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength tensor, and $\mu, \nu \in \{0, 1\}$. In the temporal gauge, $A_0 = 0$, the continuum Hamiltonian reads

$$H = \int \mathrm{d}x \left[-i\bar{\psi}\gamma^1 \left(\partial_1 + igA_1 \right) \psi + m\bar{\psi}\psi + \frac{1}{2} E^2 \right], \tag{2}$$

where the electric field E has only one component in one spatial dimension, $E = F^{10} = -\dot{A}^1$. The gamma matrices are given by

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3}$$

Using the lattice electric field operator, $E(x) \to gL(n)$, and the lattice gauge interaction in a compact formulation,[1] $e^{i\theta(n)} = e^{-iagA_1(n)}$ with $\theta(n) \in (0, 2\pi]$, the lattice Hamiltonian reads

$$H = -ix \sum_{n=0}^{N-1} [\phi^\dagger(n) e^{i\theta(n)} \phi(n+1) - \mathrm{h.c.}] + \mu \sum_{n=0}^{N-1} (-1)^n \phi^\dagger(n)\phi(n) + \sum_{n=0}^{N-1} L^2(n), \tag{4}$$

where $x = 1/(g^2 a^2)$, $\mu = 2m/(ga^2)$, and the lattice sites are labeled by $n = 0, 1, ..., N-1$, with N even. Note that the Hamiltonian in Eq. (4) was rescaled, $H \to (2/ag^2)H$, to become dimensionless, and the fermions were staggered, as discussed in the lecture, using a single-component fermion field $\phi(n)$ at each site lattice n, obeying anti-commutation relations:

$$\{\phi^\dagger(n), \phi(m)\} = \delta_{mn}, \quad \{\phi(n), \phi(m)\} = 0. \tag{5}$$

[1] For details on the compact formulation, see the appendix of the lecture notes 12.

a) Show that the mass term of staggered fermions contains a factor of $(-1)^n$, i.e.,

$$\sum_{n=0,2,4,\ldots}^{N-2} \bar{\psi}(n)\psi(n) = a\sum_{n=0}^{N-1}(-1)^n\phi^\dagger(n)\phi(n),$$

where $\bar{\psi} = \psi^\dagger\gamma^0$ and $\psi(n) = \sqrt{a}\begin{pmatrix}\phi(n)\\ \phi(n+1)\end{pmatrix}$ is a two-component spinor *(1 point)*.

b) The fermionic degrees of freedom ϕ can be mapped to spin operators using a Jordan-Wigner transformation

$$\phi(n) = \left(\prod_{l<n} i\sigma_3(l)\right)\sigma^-(n)\,, \qquad \phi^\dagger(n) = \left(\prod_{l<n}[-i\sigma_3(l)]\right)\sigma^+(n)\,,$$

where the matrices σ_i are the three Pauli matrices ($\sigma_1 = X$, $\sigma_2 = Y$, $\sigma_3 = Z$), $\sigma^\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$ and the factors of i are conventional. Show that the fermionic commutation relations are preserved by this mapping *(2 points)*.

c) Show that the Hamiltonian in terms of the spin operators can be written as

$$H = -x\sum_{n=0}^{N-1}[\sigma^+(n)e^{i\theta(n)}\sigma^-(n+1) + \text{h.c.}] + \frac{\mu}{2}\sum_{n=0}^{N-1}(-1)^n\sigma_3(n) + \sum_{n=0}^{N-1}L^2(n),$$

starting from Eq. (4) and using the Jordan-Wigner transformation *(3 points)*.

d) To ensure gauge invariance, the model needs to satisfy the lattice version of Gauss' law[2],

$$L(n) - L(n-1) = q(n),$$

where the fermionic charge is given by $q(n) = \phi^\dagger(n)\phi(n) - \frac{1}{2}[1-(-1)^n]$ for staggered fermions. Using Gauss' law and the transformation

$$\sigma^-(n) = \left(\prod_{l<n} e^{-i\theta(l)}\right)s^-(n)\,, \qquad \sigma^+(n) = \left(\prod_{l<n} e^{i\theta(l)}\right)s^+(n),$$

show that the gauge field can be completely eliminated from the dynamics assuming open boundary conditions, $L(0) = L(N-1) = 0$, and that the Hamiltonian reduces to

$$H = -x\sum_{n=0}^{N-1}[s^+(n)s^-(n+1) + \text{h.c.}] + \frac{\mu}{2}\sum_{n=0}^{N-1}(-1)^n\sigma_3(n) + \sum_{n=1}^{N-2}\left[\frac{1}{2}\sum_{m=1}^{n}(\sigma_3(m) + (-1)^m)\right]^2,$$

which contains a long-range interaction between the spins *(3 points)*.

e) What happens to the gauge field if we employ periodic boundary conditions, $L(N) = L(0)$, instead of open boundary conditions *(1 point)*?

[2]For details on the discretized Gauss' law, see the appendix of the lecture notes 12.