Exercises for Introduction to Quantum Computing (physics7514) Wise 2025

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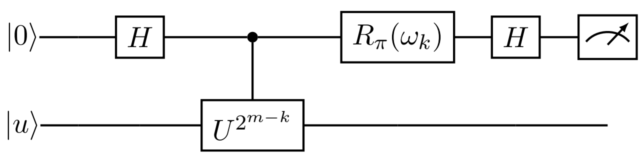
1 Handin format

Note that starting with this sheet, we will only accept **handwritten notes** for the analytical parts of the exercises. Due to the amount of LLM inspired Latex hand-ins we see ourselves forced to introduce this measure, in order to ensure a proper preparation for the exam at the end of the term. In addition we also ask you to shortly **explain your implementations** of the numerical parts moving forward.

2 Iterative Phase Estimation (IPE)

When discussing the implementation of Shor's algorithm on NISQ devices in the lecture, the question came up how to factor $N = 21$ [arXiv:1111.4147] using IPE. The original (non-iterative) phase estimation algorithm studied in the lecture is limited by the number of qubits necessary for the algorithm's precision. The IPE algorithm implements quantum phase estimation with only one single auxiliary qubit, and the accuracy of the algorithm is restricted by the number of iterations rather than the number of qubits in the first register.

Let us consider the following quantum circuit:



where $R_\pi(\omega_k) = \mathrm{diag}\left\{1, e^{-\pi i \omega_k}\right\}$ and $|u\rangle$ is an eigenstate of the operator U with eigenvalue $e^{2\pi i \phi}$ with $0 \leq \phi \leq 1$. We can write the phase as $\phi = \hat{\phi} + \delta/2^m$, where $\hat{\phi} = 0.\phi_1\phi_2...\phi_m$ is the m-bit approximation of ϕ and $0 \leq \delta < 1$ is the residual error (using the same notation as on exercise sheet 5, $0.\phi_1\phi_2...\phi_m = \frac{\phi_1}{2} + \frac{\phi_2}{2^2} + ... + \frac{\phi_m}{2^m}$).

We start running the above quantum circuit for $k = 1$ and $\omega_k = 0$, and we denote the outcome of the measurement in the computational basis by x_m, thus $x_m = 0$ or 1. Next, we run the circuit a second time with $k = 2$ and $\omega_2 = 0.x_m = \frac{x_m}{2}$, and we measure x_{m-1}. Then, we run the circuit a third time with $k = 3$ and $\omega_3 = 0.x_{m-1}x_m$, and so on, up to $k = m$. The value of ω_k at each iteration is

$$\begin{aligned}
\omega_1 &= 0\,, \\
\omega_2 &= 0.x_m = \frac{x_m}{2}\,, \\
\omega_3 &= 0.x_{m-1}x_m = \frac{x_{m-1}}{2} + \frac{x_m}{2^2}\,, \\
&\vdots \\
\omega_m &= 0.x_2 ... x_m = \frac{x_2}{2} + \frac{x_3}{2^2} + ... + \frac{x_m}{2^{m-1}}\,,
\end{aligned} \tag{1}$$

while the outcome of the measurement at each iteration is x_{m+1-k}. This is called the IPE algorithm.

a) After the first iteration $k = 1$, compute the probability of measuring ϕ_m correctly (i.e., the probability that the outcome of the measurement is $x_m = \phi_m$), as a function of δ. *(2 point).*

b) Assuming that ϕ_m was measured correctly after the first iteration, what is the probability of measuring $x_{m-1} = \phi_{m-1}$ for $k = 2$? *(2 point).*

c) After the final iteration with $k = m$, compute the probability $P(\delta)$ of measuring all the bits of $\hat{\phi} = 0.\phi_1\phi_2...\phi_m$ correctly.
Hint:

$$\prod_{k=1}^{m} \cos^2\left(\frac{\alpha}{2^k}\right) = \frac{\sin^2(\alpha)}{2^{2m}\sin^2(2^{-m}\alpha)} \tag{2}$$

(2 point).

d) What is the probability of measuring $\hat{\phi}' = \hat{\phi} + 2^{-m} = 0.\phi_1...\phi_m + 2^{-m}$ after the final iteration? *(1 point).*

e) Compute the probability $P_0(\delta)$ of measuring a phase ω_m which differs from ϕ by less than 2^{-m}. *(1 point).*

f) Like P, P_0 is monotonically decreasing in m. In the limit of $m \to \infty$, its minimum is reached for $\delta = 1/2$. Compute the minimum of the probability $\lim_{m\to\infty} P_0(\delta = 1/2)$. *(2 point).*