Exercises for Introduction to Quantum Computing (physics7514) Wise 2025

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1 Trotter Formula

Consider two Hermitian operators A, B.

a) Prove the general Trotter formula that we encountered in the lecture:

$$\lim_{n\to\infty}\left(e^{iAt/n}e^{iBt/n}\right)^n = e^{i(A+B)t}.$$

Note that this equality is true independently of whether A and B commute *(2 points)*.

b) Now, consider the case where A and B do not commute. For a discrete time evolution with time steps δt, we discussed the first-order and second-order "Suzuki-Trotter" approximations in the lecture.
Prove the first-order Suzuki-Trotter approximation:

$$S_1(\delta t) = e^{iA\delta t}\cdot e^{iB\delta t} = e^{i(A+B)\delta t} + \mathcal{O}(\delta t^2)$$

(1 point).

c) Prove the second-order Suzuki-Trotter approximation

$$S_2(\delta t) = e^{iA\delta t/2}\cdot e^{iB\delta t}\cdot e^{iA\delta t/2} = e^{i(A+B)\delta t} + \mathcal{O}(\delta t^3)$$

(1 point).

d) Find the value of $s \in \mathbb{R}$ such that the fourth-order Suzuki-Trotter approximation reads

$$S_4(\delta t) = S_2(s\delta t)S_2((1-2s)\delta t)S_2(s\delta t) = e^{i(A+B)\delta t} + \mathcal{O}(\delta t^5).$$

Hints: (1) Due to the symmetry of the formula for $S_4(\delta t)$, the terms $\mathcal{O}(\delta t^2)$ and $\mathcal{O}(\delta t^4)$ are zero. (2) It could be useful to notice that there must exist an operator R_3 such that

$$S_2(\delta t) = e^{i(A+B)\delta t + R_3\delta t^3 + \mathcal{O}(\delta t^4)} = e^{i(A+B)\delta t}e^{R_3\delta t^3} + \mathcal{O}(\delta t^4).$$

(3) Notice that up to $\mathcal{O}(\delta t^4)$, all operators in the exponent of $S_4(\delta t)$ commute *(2 points)*.

2 Quantum simulation

Consider the following Hamiltonian operator H (where the notation should not be confused with the Hadamard gate H):

$$H = \alpha Z + \beta X,$$

where α and $\beta \in \mathbb{R}$.

a) Find the eigenvalues of the operator H *(1 point).*

b) Find the values of α and β for which the operator H is unitary *(1 point).*

c) Write down an exact implementation of the operator e^{iHt}. You can write down either all the matrix elements of the resulting matrix or a linear combination of Pauli matrices and the identity *(2 points).*