Fermion-Doubling-Lecture-Notes

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Remarks

The gray derivations and remarks in these lecture notes are provided for the interested student.

1 Introduction

In the last lecture and in the exercises we derived the (quantized) Hamiltonian of Yang-Mills theory including fermions

$$H = \int \mathrm{d}^d x \Psi^\dagger \left[-\mathrm{i}\gamma^0\gamma^j\partial_j - g\gamma^0\gamma^j A_j + m\gamma^0\right]\Psi$$
$$+ \int \mathrm{d}^d x \left[\frac{1}{2}\vec{B}^a\cdot\vec{B}^a + \frac{1}{2}\vec{E}^a\cdot\vec{E}^a\right],$$

from the corresponding Lagrangian density

$$\mathcal{L}\left(\Psi,\overline{\Psi},A^a,x\right) = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a + \overline{\Psi}_l\left(i\partial\!\!\!/ + gA\!\!\!/ - m\right)\Psi. \tag{1}$$

In the Hamiltonian formalism we have thus far shown the discretization procedure for the color-electric $\vec{E}^a$ and color-magnetic fields $\vec{B}^a$.

In this lecture we will cover the challenges of discretizing the fermionic fields $\Psi, \Psi^\dagger$. For this we will consider the Hamiltonian density and corresponding Lagrangian density of free fermions in d spatial dimensions. We keep the dimensions here general because the doubling problem we will encounter later on is dimension dependent:[1]

$$\mathcal{H}_{\text{Free}}\left(\Psi,\Psi^\dagger,x\right) = \Psi^\dagger\left[-\mathrm{i}\gamma^0\gamma^j\partial_j + m\gamma^0\right]\Psi, \tag{2}$$
$$\mathcal{L}_{\text{Free}}\left(\Psi,\overline{\Psi},x\right) = \overline{\Psi}\left(i\gamma^\mu\partial_\mu - m\right)\Psi, \tag{3}$$
$$\mathcal{H}_{\text{Free}} = \mathrm{i}\Psi^\dagger\partial_0\Psi - \mathcal{L}_{\text{Free}}$$

[1] Remember that $\pi_\Psi = \frac{\delta\mathcal{L}}{\delta(\partial_0\Psi)} = \mathrm{i}\Psi^\dagger$, since we used right Grassmann derivatives instead of the usual left Grassmann derivatives in order for the quantization to proceed in a simple way without having to adjust for "Grassmann aware" Poisson brackets as in physics.stackexchange.com.

In the Hamiltonian formalism we only discretize space and not time. The naively discretized version of Eq. 2 reads

$$\begin{aligned}
&\mathcal{H}_{\text{Free}}(n,t) \\
&= \sum_{l\in\Lambda} \Psi^\dagger_\alpha(n,t) \left(-\mathrm{i} \sum_{j=1}^{d} \left(\gamma^0\gamma^j\right)_{\alpha\beta} \frac{\delta_{n+\widehat{j},l} - \delta_{n-\widehat{j},l}}{2a} + m \left(\gamma^0\right)_{\alpha\beta} \delta_{n,l} \right) \Psi_\beta(l,t).
\end{aligned} \tag{4}$$

Where

$$n,l \in \Lambda = \left\{ (n_1,\ldots,n_d) \middle| n_i \in \{0,\ldots,N_i-1\} \right\}, \tag{5}$$

$t\in\mathbb{R}$ is time, a is the spacing of the introduced lattice Λ, and $\{N_i\}_{i=1}^d$ are the spatial extents of Λ in units of a. The sum over α and β is implicit and $\widehat{j}$ denotes the unit vector in the j^{th} spatial direction. This discretization is the spatial analogue to the Euclidean version covered in the first half of the lecture and also shown by Gattringer and Lang [1, Sect. (2.2)].

2 Fermion doubling problem

The naive discretization in Eq. 4 has the well known problem of fermion doubling. By discretizing space in $\mathcal{L}_{\text{Free}}$ from Eq. 3, we can analyze the doubling here

$$\mathcal{L}_{\text{Free}}(n,t) = \int \mathrm{d}t' \sum_{l\in\Lambda} \overline{\Psi}_\alpha(n,t) D(n,l,t,t')_{\alpha,\beta} \Psi_\beta(l,t'), \tag{6}$$

where

$$\begin{aligned}
&D(n,l,t,t')_{\alpha,\beta} \\
&= \delta\left(t-t'\right) \left(\mathrm{i} \sum_{j=1}^{d} \left(\gamma^j\right)_{\alpha\beta} \frac{\delta_{n+\widehat{j},l} - \delta_{n-\widehat{j},l}}{2a} + \mathrm{i} \left(\gamma^0\right)_{\alpha,\beta} \delta_{n,l} \frac{\partial}{\partial t'} - m\delta_{\alpha,\beta}\delta_{n,l} \right)
\end{aligned} \tag{7}$$

is the naively discretized Dirac operator. In order to identify the doublers we have to consider the poles of the two point function. As you have shown in exercise sheet 3 the two point function is given by

$$\left\langle \Psi_\alpha(n,t) \overline{\Psi}_\beta(l,t') \right\rangle = D^{-1}(n,l,t,t')_{\alpha,\beta}.$$

This inversion of the naively discretized Dirac operator can easily be performed in momentum space. It should be noted that the doubling can also be identified within the Hamiltonian formalism by inspecting the equations of motion. We continue by performing a Fourier transform in t,t' and a discrete Fourier transform in n,l. By denoting $|\Lambda|$ as the number of sites in the lattice Λ, we can write:

$$D\left(\vec{p},\vec{q},p^0,q^0\right)_{\alpha,\beta} \tag{8}$$

$$
\begin{aligned}
&= \int \mathrm{d}t \int \mathrm{d}t' \frac{1}{|\Lambda|} \sum_{n,l\in\Lambda} \mathrm{e}^{-\mathrm{i}\left(p^0 t - \vec{p}\cdot\vec{n}a\right)} D(n,l,t,t')_{\alpha,\beta} \mathrm{e}^{+\mathrm{i}\left(q^0 t' - \vec{q}\cdot\vec{l}a\right)} \\
&= \int \mathrm{d}t \frac{1}{|\Lambda|} \sum_{n\in\Lambda} \mathrm{e}^{-\mathrm{i}t\left(p^0 - q^0\right)} \mathrm{e}^{+\mathrm{i}\vec{p}\cdot\vec{n}a} \\
&\cdot \left[\mathrm{i} \sum_{j=1}^{d} \left(\gamma^j\right)_{\alpha\beta} \frac{1}{2a} \left(\mathrm{e}^{-\mathrm{i}\vec{q}\cdot\overrightarrow{(n+\widehat{j})}} - \mathrm{e}^{-\mathrm{i}\vec{q}\cdot\overrightarrow{(n-\widehat{j})}} \right) + \left(\mathrm{i}\left(\gamma^0\right)_{\alpha,\beta} \mathrm{i} q^0 - m\delta_{\alpha,\beta} \right) \mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{n}a} \right] \\
&= 2\pi\delta\left(p^0 - q^0\right) \frac{1}{|\Lambda|} \sum_{n\in\Lambda} \mathrm{e}^{+\mathrm{i}(\vec{p}-\vec{q})\cdot\vec{n}a} \\
&\cdot \left[\mathrm{i} \sum_{j=1}^{d} \left(\gamma^j\right)_{\alpha\beta} \frac{1}{2a} \left(\mathrm{e}^{-\mathrm{i}q^j a} - \mathrm{e}^{+\mathrm{i}q^j a} \right) - \left(\gamma^0\right)_{\alpha,\beta} q^0 - m\delta_{\alpha,\beta} \right] \\
&= 2\pi\delta\left(p^0 - q^0\right) \delta_{\vec{p},\vec{q}} \cdot \left[\sum_{j=1}^{d} \left(\gamma^j\right)_{\alpha\beta} \frac{-1}{a} \frac{1}{2\mathrm{i}} \left(\mathrm{e}^{\mathrm{i}q_j a} - \mathrm{e}^{-\mathrm{i}q_j a} \right) - \left(\gamma^0\right)_{\alpha,\beta} q^0 - m\delta_{\alpha,\beta} \right] \\
&= 2\pi\delta\left(p^0 - q^0\right) \delta_{\vec{p},\vec{q}} \underbrace{\left[-\sum_{j=1}^{d} \left(\gamma^j\right)_{\alpha\beta} \frac{1}{a} \sin\left(p_j a\right) - \left(\gamma^0\right)_{\alpha,\beta} p^0 - m\delta_{\alpha,\beta} \right]}_{=:\tilde{D}(p)_{\alpha,\beta}}.
\end{aligned}
$$

Where $p^0, q^0 \in \mathbb{R}$ and

$$
\vec{p},\vec{q} \in \tilde{\Lambda} := \left\{ \left(\frac{2\pi}{aN_1}(k_1 + b_1), \dots, \frac{2\pi}{aN_d}(k_d + b_d) \right) \middle| k_i \in \left\{ -\left\lceil \frac{N_i}{2} \right\rceil + 1, \dots, \left\lfloor \frac{N_i}{2} \right\rfloor \right\} \right\}. \tag{9}
$$

With this we have assumed quasi periodic boundary conditions. $b_i = 0$ corresponds to periodic boundary conditions and $b_i = \frac{1}{2}$ yields anti periodic boundary conditions. From here on out we will set $\forall i : b_i = 0$ such that

$$
-\frac{\pi}{a} < p_i, q_i \leq \frac{\pi}{a}. \tag{10}
$$

Written in a more compact form we now have the momentum space Dirac operator $\tilde{D}(p)$:

$$
\begin{aligned}
\tilde{D}(p) &= -m\mathbf{1} - \gamma^0 p_0 - \sum_{j=1}^{d} \gamma^j \frac{\sin\left(p_j a\right)}{a} \\
\Rightarrow \tilde{D}^{-1}(p) &= \frac{-m\mathbf{1} + \gamma^0 p_0 + \sum_{j=1}^{d} \gamma^j \frac{\sin\left(p_j a\right)}{a}}{m^2 - \left(p_0^2 - \sum_{j=1}^{d} \left(\frac{\sin\left(p_j a\right)}{a} \right)^2 \right)}.
\end{aligned}
$$

As we will see this propagator $\tilde{D}^{-1}(p)$ has additional poles in in the continuum limit $a \to 0$.

In detail the poles are the momenta for which:

$$m^2 - \left(p_0^2 - \sum_{j=1}^{d} \left(\frac{\sin(p_j a)}{a} \right)^2 \right) \overset{!}{=} 0. \tag{11}$$

In order to analyze the solutions of this equation, we define the lattice momenta q_j

$$q_j := \frac{\sin(p_j a)}{a}. \tag{12}$$

In the continuum limit we have $q_j \in \mathbb{R}$. We want to identify every pole with a particle, for this we have to pick a reference frame. Without loss of generality we can work in the rest frame of the lattice momentum

$$(p_0, \vec{q}) = \left(m, \vec{0} \right). \tag{13}$$

In this reference frame Eq. 11 is automatically satisfied. The conditions on p_j are then:

$$\begin{aligned} 0 &\overset{13}{=} q_j \overset{12}{=} \frac{\sin(p_j a)}{a}, \\ &\overset{10}{\Leftrightarrow} p_j = 0 \vee p_j = \frac{\pi}{a}. \end{aligned} \tag{14}$$

This shows that there are in total 2^d particles, $2^d - 1$ doublers:

$$p_\mu = \left(p_0, 0 \vee \frac{\pi}{a}, \ldots, 0 \vee \frac{\pi}{a} \right)_\mu. \tag{15}$$

If one were to also discretize time the number of particles would double to 2^{d+1} since p_0 then also involves a $\sin(\cdot)$ similar to the spatial momenta. The relation between the lattice momentum q_j and the parameter p_j are also shown in Fig. 1.

There are many discretizations schemes currently on the market to tackle the fermion doubling problem. Due the Nielsen-Ninomiya theorem[2, 3] we know that any fermion discretization must violate at least one of the following properties:

- Locality
- Translational invariance
- Chiral symmetry
- Correct continuum limit

In the following we will discuss the two most common discretization choices Wilson and Staggered fermions.

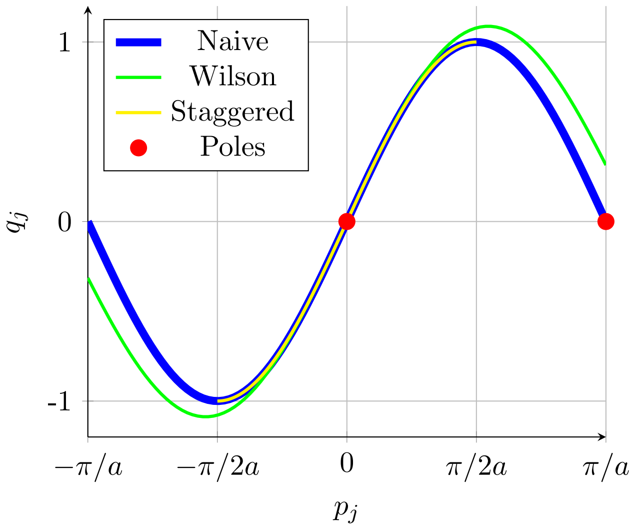

Figure 1: Illustration of the relation between the lattice momentum q_j and the parameter p_j for the naive (blue), Wilson (green) and Staggered(yellow) fermion discretizations. The poles of the naive discretization (red) are also highlighted.

3 Wilson fermions

The idea of the fermion discretization introduced by Wilson [4] is to give the unwanted doublers a mass which diverges in the continuum limit. Thus the doublers decouple from the theory and Chiral symmetry is explicitly broken. The corresponding change in the lattice momentum is also illustrated in Fig. 1. Specifically the additional poles can be counter acted by adding the following term to $\tilde{D}(p)$:

$$\tilde{\mathcal{L}}_{\text{Wilson}}(\vec{p}) = -\frac{r}{a}\mathbf{1}\sum_{j=1}^{d}\left[1-\cos\left(p_j a\right)\right]. \tag{16}$$

Where $r \in \mathbb{R}$ is the Wilson parameter. The Euclidean version of this term was briefly mentioned in the first half of the lecture, for a detailed discussion of that version see Gattringer and Lang [1, Sect. (5.2.3)]. The new momentum space Dirac operator $\tilde{D}_{\text{Wilson}}(p)$ is then:

$$\tilde{D}_{\text{Wilson}}(p) = -\left(\frac{r}{a}\sum_{j=1}^{d}\left[1-\cos\left(p_j a\right)\right]+m\right)\mathbf{1} - \gamma^0 p_0 - \sum_{j=1}^{d}\gamma^j\frac{\sin\left(p_j a\right)}{a}, \tag{17}$$

$$\Rightarrow \tilde{D}^{-1}(p) = \frac{-\left(\frac{r}{a}\sum_{j=1}^{d}\left[1-\cos\left(p_j a\right)\right]+m\right)\mathbf{1} + \gamma^0 p_0 + \sum_{j=1}^{d}\gamma^j\frac{\sin\left(p_j a\right)}{a}}{\left(\frac{r}{a}\sum_{j=1}^{d}\left[1-\cos\left(p_j a\right)\right]+m\right)^2 - \left(p_0^2 - \sum_{j=1}^{d}\left(\frac{\sin\left(p_j a\right)}{a}\right)^2\right)}. \tag{18}$$

The poles are now the momenta for which:

$$\left(\frac{r}{a}\sum_{j=1}^{d}\left[1-\cos\left(p_j a\right)\right]+m\right)^2-\left(p_0^2-\sum_{j=1}^{d}\left(\frac{\sin\left(p_j a\right)}{a}\right)^2\right)\overset{!}{=}0. \tag{19}$$

For $p_j < -\frac{\pi}{2a}$ or $p_j > \frac{\pi}{2a}$ we have that

$$\cos\left(p_j a\right)=-\sqrt{1-\sin^2\left(p_j a\right)}. \tag{20}$$

In this case the rest frame of the lattice momentum results in:

$$\begin{aligned}
&\left(\frac{r}{a}\sum_{j=1}^{d}\left[1-\cos\left(p_j a\right)\right]+m\right)^2-\left(p_0^2-\sum_{j=1}^{d}\left(\frac{\sin\left(p_j a\right)}{a}\right)^2\right)\\
\overset{20}{=}&\left(\frac{r}{a}\sum_{j=1}^{d}\left[1+\sqrt{1-\sin^2\left(p_j a\right)}\right]+m\right)^2-\left(p_0^2-\sum_{j=1}^{d}\left(\frac{\sin\left(p_j a\right)}{a}\right)^2\right)\\
\overset{12}{=}&\left(\frac{r}{a}\sum_{j=1}^{d}\left[1+\sqrt{1-a^2q_j^2}\right]+m\right)^2-\left(p_0^2-\sum_{j=1}^{d}q_j^2\right)\\
\overset{13}{=}&\left(\frac{2dr}{a}+m\right)^2-m^2\neq 0.
\end{aligned}$$

For $-\frac{\pi}{2a}\leq p_j\leq\frac{\pi}{2a}$ we have that

$$\cos\left(p_j a\right)=\sqrt{1-\sin^2\left(p_j a\right)}. \tag{21}$$

In this case the rest frame of the lattice momentum results in:

$$\begin{aligned}
&\left(\frac{r}{a}\sum_{j=1}^{d}\left[1-\cos\left(p_j a\right)\right]+m\right)^2-\left(p_0^2-\sum_{j=1}^{d}\left(\frac{\sin\left(p_j a\right)}{a}\right)^2\right)\\
\overset{21}{=}&\left(\frac{r}{a}\sum_{j=1}^{d}\left[1-\sqrt{1-\sin^2\left(p_j a\right)}\right]+m\right)^2-\left(p_0^2-\sum_{j=1}^{d}\left(\frac{\sin\left(p_j a\right)}{a}\right)^2\right)\\
\overset{12}{=}&\left(\frac{r}{a}\sum_{j=1}^{d}\left[1-\sqrt{1-a^2q_j^2}\right]+m\right)^2-\left(p_0^2-\sum_{j=1}^{d}q_j^2\right)\\
\overset{13}{=}&\,m^2-m^2=0.
\end{aligned}$$

We see that in the continuum limit only the physical pole

$$p_\mu=\left(p_0,0,\ldots,0\right)_\mu \tag{22}$$

remains.

We have to add the corresponding term to the Hamiltonian density:

$$\tilde{\mathcal{H}}_{\text{Wilson}}\left(\vec{p}\right) = -\tilde{\mathcal{L}}_{\text{Wilson}}\left(\vec{p}\right) = \frac{r}{a}\mathbf{1}\sum_{j=1}^{d}\left[1-\cos\left(p_j a\right)\right], \tag{23}$$

$$\mathcal{H}_{\text{Wilson}}\left(\vec{p},\vec{q},p^0,q^0\right) = 2\pi\delta\left(p^0-q^0\right)\delta_{\vec{p},\vec{q}}\tilde{\mathcal{H}}_{\text{Wilson}}\left(\vec{p}\right). \tag{24}$$

Where the additional factors in $\mathcal{H}_{\text{Wilson}}\left(\vec{p},\vec{q},p^0,q^0\right)$ are introduced in analogy to Eq. 8. Fourier transforming back we obtain:

$$\begin{aligned}
&\mathcal{H}_{\text{Wilson}}\left(n,m,t,t'\right) \\
&= \int\frac{\mathrm{d}p^0}{2\pi}\int\frac{\mathrm{d}q^0}{2\pi}\frac{1}{\left|\tilde{\Lambda}\right|}\sum_{p,q\in\tilde{\Lambda}}\mathrm{e}^{+\mathrm{i}\left(p^0t-\vec{p}\cdot\vec{n}a\right)}\mathcal{H}_{\text{Wilson}}\left(\vec{p},\vec{q},p^0,q^0\right)\mathrm{e}^{-\mathrm{i}\left(q^0t'-\vec{q}\cdot\vec{m}a\right)} \\
&\overset{24}{=} \int\frac{\mathrm{d}p^0}{2\pi}\mathrm{e}^{\mathrm{i}p^0(t-t')}\frac{1}{\left|\tilde{\Lambda}\right|}\sum_{p\in\tilde{\Lambda}}\mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{n}-\vec{m})a}\tilde{\mathcal{H}}_{\text{Wilson}}\left(\vec{p}\right) \\
&\overset{23}{=} ra\mathbf{1}\delta\left(t-t'\right)\sum_{j=1}^{d}\frac{1}{2a^2}\left[\frac{1}{\left|\tilde{\Lambda}\right|}\sum_{p\in\tilde{\Lambda}}\left(2-2\cos\left(p_ja\right)\right)\mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{n}-\vec{m})a}\right] \\
&= ra\mathbf{1}\delta\left(t-t'\right)\sum_{j=1}^{d}\frac{1}{2a^2}\left[2\delta_{n,m}-\frac{1}{\left|\tilde{\Lambda}\right|}\sum_{p\in\tilde{\Lambda}}\left(\mathrm{e}^{\mathrm{i}\overbrace{p_ja}^{-p^ja}}+\mathrm{e}^{-\mathrm{i}\overbrace{p_ja}^{-p^ja}}\right)\mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{n}-\vec{m})a}\right] \\
&= ra\mathbf{1}\delta\left(t-t'\right)\sum_{j=1}^{d}\frac{1}{2a^2}\left[2\delta_{n,m}-\frac{1}{\left|\tilde{\Lambda}\right|}\sum_{p\in\tilde{\Lambda}}\left(\mathrm{e}^{-\mathrm{i}\vec{p}\cdot\overrightarrow{(n-m+\hat{j})}a}+\mathrm{e}^{-\mathrm{i}\vec{p}\cdot\overrightarrow{(n-m-\hat{j})}a}\right)\right] \\
&= ra\mathbf{1}\delta\left(t-t'\right)\sum_{j=1}^{d}\frac{1}{2a^2}\left[2\delta_{n,m}-\delta_{n+\hat{j},m}-\delta_{n-\hat{j},m}\right].
\end{aligned} \tag{25}$$

In total we have to add the following term to the Hamiltonian density:

$$\begin{aligned}
\mathcal{H}_{\text{Wilson,tot}}(n,t) &= \int\mathrm{d}t'\sum_{m\in\Lambda}\overline{\Psi}(n,t)\mathcal{H}_{\text{Wilson}}\left(n,m,t,t'\right)\Psi(m,t') \\
&\overset{25}{=}\sum_{m\in\Lambda}\left[\overline{\Psi}_\alpha(n,t)\left(ra\delta_{\alpha,\beta}\sum_{j=1}^{d}\frac{1}{2a^2}\left[2\delta_{n,m}-\delta_{n+\hat{j},m}-\delta_{n-\hat{j},m}\right]\right)\Psi_\beta(m,t)\right].
\end{aligned} \tag{26}$$

4 Staggered fermions

The additional poles can also be (partially) removed via the so-called staggering procedure. For this we first have to remind ourselves of the dimensionality v of the irreducible representation of the Clifford algebra (number of spinor components) in $(d+1)$ spacetime dimensions. According to Weinberg [5, Sect. (5.4)] we have

$$v = \begin{cases} 2^{\frac{d}{2}} & d \text{ even} \\ 2^{\frac{d+1}{2}} & d \text{ odd} \end{cases}.$$

The following discussion is inspired by the Euclidean version presented by Rothe [6, Sect. (4.4)].

The idea of the fermion discretization introduced by Kogut and Susskind [7] is to reduce the Brillouin zone by doubling the effective lattice spacing. To achieve this we want to distribute the fermionic degrees of freedom over a unit cell (d dimensional hypercube of side length a) thus doubling the effective lattice spacing.

There are 2^d many sites in a hypercube and only v many components of a Dirac field. Thus we need

$$\begin{cases} 2^{\frac{d}{2}} & d \text{ even} \\ 2^{\frac{d-1}{2}} & d \text{ odd} \end{cases}$$

many different Dirac fields to cover the hypercube and reduce the Brillouin zone by a factor of 2 in each direction. Note that this restricts us to the invertible region of the $\sin(\cdot)$ encountered in the Doubling problem as also shown in Fig. 1.

The realization of this idea turns out to be more complicated. The sites of an elementary hypercube will be occupied by linear combinations of different flavors of Dirac fields Ψ_α^f in order to obtain the desired continuum limit.

After this motivation we can now move onto the actual derivation. For this we can first write Eq. 6 as

$$\mathcal{L}_{\text{Free}}(n,t) = \int \mathrm{d}t' \sum_{m\in\Lambda} \Psi_\alpha^\dagger(n,t)\gamma^0_{\alpha,\beta} D(n,m,t,t')_{\beta,\gamma}\Psi_\gamma(m,t').$$

We want to redistribute the fermionic degrees of freedom within a unit cell in such a way that the resulting expression is diagonal in spinor space. Once this is achieved we will keep only one component to arrive at the desired number of degrees of freedom distributed within a unit cell.

The goal is now to diagonalize this expression in spinor space via a *local* change of variables. We will thus intertwine spinor and spatial indices via the following transformation[2]

$$\Psi(n) = T(n)\chi(n), \qquad \Psi^\dagger(n) = \chi^\dagger(n)T^\dagger(n).$$

The diagonalization conditions then read

$$T^\dagger(n)\gamma^0\gamma^j T(n+\widehat{j}) \stackrel{!}{=} \eta_j^+(n)\mathbf{1}, \tag{27}$$

$$T^\dagger(n)\gamma^0\gamma^j T(n-\widehat{j}) \stackrel{!}{=} \eta_j^-(n)\mathbf{1}, \tag{28}$$

[2] In the Lagrangian version we would transform Ψ and $\overline{\Psi}$ instead. And in that case the *discretized* imaginary time component would result in a similar condition to the spatial ones.

$$T^\dagger(n)T(n) \stackrel{!}{=} \eta_0(n)\mathbf{1}, \tag{29}$$

$$T^\dagger(n)\gamma^0 T(n) \stackrel{!}{=} \varepsilon(n)\mathbf{1}. \tag{30}$$

Before we attempt to find a $T(n)$ which satisfies these conditions we should pay special attention to Eq. 30. Since

$$\left(\gamma^0\right)^2 = \mathbf{1}$$

we know that γ^0 only has eigenvalues ± 1. In addition these eigenvalues appear with equal multiplicity $d_+ = d_- \neq 0$ because

$$0 = \operatorname{tr}\left(\gamma^0\right) = d_+ - d_-, \qquad d_+ + d_- = v$$

Thus Eq. 30 can *not* be satisfied. As you will show in the exercises

$$T(n) = \left(\gamma^0\gamma^1\right)^{n_1} \dots \left(\gamma^0\gamma^d\right)^{n_d}$$

solves Eq. 27, Eq. 28 and Eq. 29 with

$$\eta_0(n) = 1, \qquad \eta_j(n) \coloneqq \eta_j^+(n) = \eta_j^-(n) = (-1)^{\sum_{i=1}^{j-1} n_i}.$$

With this choice we obtain

$$T^\dagger(n)\gamma^0 T(n) = (-1)^{\sum_{i=1}^{d} n_i}\gamma^0 \eqqcolon \tilde{\varepsilon}(n)\gamma^0$$

instead of Eq. 30. Such that Eq. 6 can now be expressed as

$$\mathcal{L}_{\text{Free}}(n,t) = \chi^\dagger(n,t)\Bigg\{\mathrm{i}\sum_{j=1}^{d}\eta_j(n)\frac{1}{2a}\left[\chi(n+\hat{j},t) - \chi(n-\hat{j},t)\right] \\ + \mathrm{i}\partial_t\chi(n,t) - m\tilde{\varepsilon}(n)\gamma^0\chi(n,t)\Bigg\}.$$

And Eq. 4 can similarly be expressed as

$$\mathcal{H}_{\text{Free}}(n,t) = \chi^\dagger(n,t)\Bigg\{-\mathrm{i}\sum_{j=1}^{d}\eta_j(n)\frac{1}{2a}\left[\chi(n+\hat{j},t) - \chi(n-\hat{j},t)\right] + m\tilde{\varepsilon}(n)\gamma^0\chi(n,t)\Bigg\}.$$

We now have to consider an eigenvalue subspace of γ^0 in order to proceed[3]. For this we decompose

$$\chi(n) = \chi_+(n) + \chi_-(n), \qquad \chi_\pm = \frac{1}{2}(1 \pm \gamma^0)\chi,$$

[3] This step is only necessary in the Hamiltonian version. Thus the resulting staggered fermions correspond to reduced staggered fermions from the Lagrangian version as discussed by Catterall, Pradhan, and Samlodia [8].

and choose to set $\chi_-(n) = 0$. Thus we now have

$$\mathcal{L}_{\text{Free}}(n,t) = \chi_+^\dagger(n,t)\Bigg\{\mathrm{i}\sum_{j=1}^{d}\eta_j(n)\frac{1}{2a}\left[\chi_+(n+\widehat{j},t) - \chi_+(n-\widehat{j},t)\right] + \mathrm{i}\partial_t\chi_+(n,t) - m\tilde{\varepsilon}(n)\chi_+(n,t)\Bigg\},$$

$$\mathcal{H}_{\text{Free}}(n,t) = \chi_+^\dagger(n,t)\Bigg\{-\mathrm{i}\sum_{j=1}^{d}\eta_j(n)\frac{1}{2a}\left[\chi_+(n+\widehat{j},t) - \chi_+(n-\widehat{j},t)\right] + m\tilde{\varepsilon}(n)\chi_+(n,t)\Bigg\}.$$

At this point we have achieved our goal of a diagonal expression in spinor space. We will now only keep one component in spinor space and call it Ψ, such that the final staggered Lagrangian density is given by

$$\mathcal{L}_{\text{Stag}}(n,t) = \Psi^\dagger(n,t)\Bigg\{\mathrm{i}\sum_{j=1}^{d}\eta_j(n)\frac{1}{2a}\left[\Psi(n+\widehat{j},t) - \Psi(n-\widehat{j},t)\right] + \mathrm{i}\partial_t\Psi(n,t) - m\tilde{\varepsilon}(n)\Psi(n,t)\Bigg\}.$$

The corresponding Hamiltonian is

$$\begin{aligned}\mathcal{H}_{\text{Stag}}(n,t) &= \Psi^\dagger(n,t)\Bigg\{-\mathrm{i}\sum_{j=1}^{d}\eta_j(n)\frac{1}{2a}\left[\Psi(n+\widehat{j},t) - \Psi(n-\widehat{j},t)\right] + m\tilde{\varepsilon}(n)\Psi(n,t)\Bigg\}\\ &= \mathrm{i}\Psi^\dagger(n,t)\partial_t\Psi(n,t) - \mathcal{L}_{\text{Stag}}(n,t).\end{aligned} \tag{31}$$

In order to analyze the doublers of this resulting theory we have to express it on the relevant lattice (spacing of $2a$). For this we introduce hyper-cubic coordinates

$$\begin{aligned} n_j &= 2r_j + s_j,\\ r \in R &= \left\{(r_1,\ldots,r_d)\middle| r_i \in \left\{0,\ldots,\frac{N_i}{2}-1\right\}\right\},\\ s \in S &= \left\{(s_1,\ldots,s_d)\middle| s_i \in \{0,1\}\right\}.\end{aligned}$$

With these we can introduce a 2^d component spinor

$$\chi_s(r) \coloneqq \Psi(2r+s).$$

Further note that

$$\eta_j(n) = \eta_j(s), \qquad \tilde{\varepsilon}(n) = \tilde{\varepsilon}(s).$$

We can express $\Psi(2r+s\pm\widehat{j})$ in terms of $\chi_s(r)$ like follows

$$\Psi(2r+s\pm\widehat{j}) = \sum_{s'\in S}\left[\delta_{s\pm\widehat{j},s'}\chi_{s'}(r) + \delta_{s\mp\widehat{j},s'}\chi_{s'}(r\pm\widehat{j})\right].$$

With this the staggered Lagrangian density now reads

$$\mathcal{L}_{\text{Stag}}(n,t) = \int \mathrm{d}t' \sum_{r'\in R} \chi^\dagger(r,t) D_{\text{Stag}}(r,r',t,t')_{s,s'} \chi_{s'}(r',t'),$$

where

$$\begin{aligned}
&D_{\text{Stag}}(r,r',t,t')_{s,s'} \\
&= \delta\left(t-t'\right)\left(\frac{\mathrm{i}}{2a}\sum_{j=1}^{d}\eta_j(s)\left[\delta_{s+\widehat{j},s'}(\delta_{r,r'}-\delta_{r-\widehat{j},r'})+\delta_{s-\widehat{j},s'}(\delta_{r+\widehat{j},r'}-\delta_{r,r'})\right]\right. \\
&\qquad\qquad \left. +\,\delta_{s,s'}\delta_{r,r'}\left[\mathrm{i}\partial_{t'}-m\tilde{\varepsilon}(s)\right]\right)
\end{aligned} \tag{32}$$

is the staggered Dirac operator. As you will show in the exercises the Fourier transform of Eq. 32 is given by

$$\begin{aligned}
&D_{\text{Stag}}\left(\vec{p},\vec{q},p^0,q^0\right)_{\alpha,\beta} \\
&= \int \mathrm{d}t \int \mathrm{d}t' \frac{1}{|R|}\sum_{n,m\in R} \mathrm{e}^{-\mathrm{i}\left(p^0 t-\vec{p}\cdot\vec{n}a\right)} D_{\text{Stag}}(n,m,t,t')_{\alpha,\beta}\mathrm{e}^{+\mathrm{i}\left(q^0 t'-\vec{q}\cdot\vec{m}a\right)} \\
&= 2\pi\delta\left(p^0-q^0\right)\delta_{\vec{p},\vec{q}}\underbrace{\left(\Gamma^0\right)_{\alpha\gamma}\left[-\sum_{j=1}^{d}\left(\Gamma^j\right)_{\gamma\beta}(\vec{p})\frac{1}{a}\sin\left(p_j\frac{a}{2}\right)-\left(\Gamma^0\right)_{\gamma,\beta}p^0-m\delta_{\gamma,\beta}\right]}_{=:\tilde{D}_{\text{Stag}}(p)_{\alpha,\beta}},
\end{aligned} \tag{33}$$

where we introduced

$$\left(\Gamma^0\right)_{\alpha\beta} := \delta_{\alpha,\beta}\tilde{\varepsilon}(\alpha), \tag{34}$$

$$\left(\Gamma^j\right)_{\alpha\beta}(\vec{p}) := \mathrm{e}^{\mathrm{i}\frac{a}{2}\vec{p}\cdot(\beta-\alpha)}\eta_j(\alpha)\tilde{\varepsilon}(\alpha)\left[\delta_{\alpha+\widehat{j},\beta}+\delta_{\alpha-\widehat{j},\beta}\right]. \tag{35}$$

$\Gamma^\mu(\vec{p})$ constitute a 2^d dimensional representation of the Clifford algebra when considering equal momenta $\vec{p}$:

$$\{\Gamma^\mu(\vec{p}),\Gamma^\nu(\vec{p})\} = 2g^{\mu,\nu}\mathbf{1}.$$

This can easily be shown. We thus know that the inverse of the momentum space Dirac operator $\tilde{D}_{\text{Stag}}(p)$ is given by

$$\tilde{D}^{-1}_{\text{Stag}}(p) = \frac{-m\mathbf{1}+\Gamma^0 p_0+\sum_{j=1}^{d}\Gamma^j(\vec{p})\frac{\sin\left(p_j\frac{a}{2}\right)}{a}}{m^2-\left(p_0^2-\sum_{j=1}^{d}\left(\frac{\sin\left(p_j\frac{a}{2}\right)}{a}\right)^2\right)}\Gamma^0.$$

Which yields the same equations for the poles as in the naive case, with $p_j \to \frac{1}{2}p_j$. Such that there are no doublers left in this 2^d dimensional *reducible* representation of the Dirac Formalism $\chi_s(r)$.

One can make contact with our initial idea by decomposing this 2^d dimensional *reducible* representation into v dimensional *irreducible* representations. For this see [6, 8].

5 Adding gauge fields

Gauge fields can be introduced into Eq. 31 in the same way they were introduced in first half of the lecture. We can further make them dynamical by adding the gauge invariant lattice Hamiltonian H_g from the last lecture. The resulting lattice Hamiltonian density of Yang-Mills[4] theory including staggered fermions is then:[5]

$$\begin{aligned}\mathcal{H}(n,t) = \Psi^\dagger(n,t)\Big\{ -\mathrm{i}\sum_{j=1}^{d}\eta_j(n)\frac{1}{2a}\Big[U_{n,j}\Psi(n+\widehat{j},t) - U^\dagger_{n-\widehat{j},j}\Psi(n-\widehat{j},t)\Big] \\ + m\tilde{\varepsilon}(n)\Psi(n,t)\Big\} \\ + \frac{1}{g^2}\sum_{k,l}\Big[1-\frac{1}{2}\Big(U_{n,kl}+U^\dagger_{n,kl}\Big)\Big] + \frac{1}{2}g^2\sum_{k}L^2_{n,k}.\end{aligned}$$

And the accompanying lattice version of Gauss' law is given by

$$G_n\left|\Psi\right\rangle = 0, \qquad G_n = g\Psi^\dagger(n)\Psi(n) + \sum_{k \text{ outgoing from site } n} L_{n,k} - \sum_{k \text{ ingoing to site } n} L_{n,k}.$$

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