Exercises for Lattice Field Theory – Hamiltonian and Lagrangian Methods (physics7520)
Sose 2026

L. Funcke, S. Singh, A. Bulgarelli and N. Dichter

---

0. Install Julia

In the next exercise sheet you will use `julia` for programming a tensor network. As a preparation for this install `julia` from `julialang.org`.

1. Hamiltonian for general gauge theories including fermions

In the lecture you have seen the Hamiltonian of Yang-Mills theory of a U(1)-gauge field including fermions. In this exercise you will derive the Hamiltonian in the more general case of a compact Lie group (e.g. SU(N) or U(1)).
We will start from the classical Lagrangian density given by

$$\mathcal{L}\left(\Psi, \overline{\Psi}, A^a, x\right) = -\frac{1}{4} F_a^{\mu\nu} F^a_{\mu\nu} + \overline{\Psi}\left(\mathrm{i} D_\mu \gamma^\mu - m\right)\Psi. \tag{1}$$

Where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu,$$

is the field strength tensor and

$$D_\mu = \partial_\mu - \mathrm{i} g A^a_\mu(x) t^a.$$

is the covariant derivative. Here $A^a_\mu(x)$ are now the generalized gauge fields with the index a labeling the so-called adjoint representation (for SU(N) we have $a \in \{1, \ldots, N^2 - 1\}$). t^a are the generators in the fundamental representation of the Lie algebra corresponding to the compact Lie group under consideration (for SU(N) there are $N^2 - 1$ of them with each of them being represented as a $N \times N$ matrix in the fundamental representation). Thus, we are actually considering a vector of spinors Ψ_l ($l \in \{1, \ldots, N\}$ for SU(N)). These generators are traceless, hermitian and satisfy

$$\left[t^a, t^b\right] = i f^{abc} t^c,$$
$$\mathrm{tr}\left(t^a t^b\right) = \frac{1}{2}\delta_{a,b},$$

where f^{abc} are the (antisymmetric) structure constants and the trace condition fixes a normalization.

a) *(2 pts.)* In order to derive the classical Hamiltonian density, you first have to calculate the canonical momenta:[1]

$$
\begin{aligned}
\pi_\Psi^l &= \frac{\delta\mathcal{L}}{\delta(\partial_0\Psi_l)} \\
\pi_A^{a,0} &= \frac{\delta\mathcal{L}}{\delta(\partial_0 A_0^a)} \\
\pi_A^{a,k} &= \frac{\delta\mathcal{L}}{\delta(\partial_0 A_k^a)}
\end{aligned}
$$

b) *(6 pts.)* Using the results obtained in the previous part calculate the full expression for the classical Hamiltonian density

$$\mathcal{H} = \pi_\Psi^l(\partial_0\Psi_l) + \pi_A^{a,k}(\partial_0 A_k^a) - \mathcal{L}.$$

c) *(4 pts.)* We now consider the full *classical* Hamiltonian in d spatial dimensions

$$H = \int \mathrm{d}^d x \mathcal{H}.$$

Insert the result you obtained in the previous part and perform a suitable integration by parts (dropping surface terms[2]) to arrive at

$$
\begin{aligned}
H = &\int \mathrm{d}^d x \Psi^\dagger \left[-\mathrm{i}\gamma^0\gamma^j\partial_j - g\gamma^0\gamma^j A_j + m\gamma^0\right]\Psi \\
&+ \int \mathrm{d}^d x \left[\frac{1}{4}F_a^{kl}F_{kl}^a - \frac{1}{2}\pi_A^{a,k}\pi_{A,k}^a\right] \\
&- \int \mathrm{d}^d x A_0^a \left[\partial_k F_a^{k0} + g\Psi^\dagger t^a\Psi + g f^{abc}A_k^b F_c^{k0}\right].
\end{aligned}
$$

d) *(2 pts.)* Explain how the final expression from the previous part changes in the case of an abelian gauge theory (e.g. U(1)) such that you arrive at the classical Hamiltonian given in the lecture.

e) *(2 pts.)* After the quantization and the necessary adoption of the temporal gauge we arrive at the *quantized* Hamiltonian

$$
\begin{aligned}
H = &\int \mathrm{d}^d x \Psi^\dagger \left[-\mathrm{i}\gamma^0\gamma^j\partial_j - g\gamma^0\gamma^j A_j + m\gamma^0\right]\Psi \\
&+ \int \mathrm{d}^d x \left[\frac{1}{4}F_a^{kl}F_{kl}^a - \frac{1}{2}\pi_A^{a,k}\pi_{A,k}^a\right],
\end{aligned}
$$

together with the Gauss law

$$G^a\left|\Psi\right\rangle = 0, \qquad G^a = g\Psi^\dagger t^a\Psi + \partial_\mu F_a^{\mu 0} - g f^{abc}F_b^{\nu 0}A_\nu^c.$$

---

[1] Note that we here use right Grassmann derivatives $\left(\frac{\partial_R \eta_1\eta_2}{\partial\eta_2} = +\eta_1\right)$ instead of the usual left Grassmann derivatives $\left(\frac{\partial_L \eta_1\eta_2}{\partial\eta_2} = -\eta_1\right)$ such that the following quantization proceeds in a simple way without having to adjust for "Grassmann aware" Poisson brackets as in `physics.stackexchange.com`.

[2] Dropping these surface terms is actually fine here since they are proportional to A_0^a and thus vanish in the following gauge fixing.

Express the pure gauge part of this Hamiltonian

$$H_{\mathrm{YM}} = \int_{\mathcal{M}} \mathrm{d}^d x \left[\frac{1}{4} F_a^{kl} F_{kl}^a - \frac{1}{2} \pi_A^{a,k} \pi_{A,k}^a \right],$$

in terms of color-electric[3] and color-magnetic fields

$$\begin{aligned} E_k^a &= F_{0k}^a = \partial_0 A_k^a, \\ B_j^a &= -\frac{1}{2} \varepsilon_{jkl} F_{kl}. \end{aligned}$$

2. Staggering transformation

In this exercise you will show that the transformation

$$T(n) = \left(\gamma^0 \gamma^1\right)^{n_1} \dots \left(\gamma^0 \gamma^d\right)^{n_d}$$

given in the lecture indeed satisfies most of the conditions mentioned there.

a) *(2 pts.)* By making use of

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \mathbf{1}, \qquad (\gamma^\mu)^\dagger = \gamma^0 \gamma^\mu \gamma^0,$$

show that $\gamma^0 \gamma^j$ is hermitian and

$$\left\{\gamma^0 \gamma^i, \gamma^0 \gamma^j\right\} \stackrel{!}{=} 2\delta_{i,j} \mathbf{1}$$

b) *(2 pts.)* Using the results from the previous part show that $T(n)$ is unitary and

$$T(n + \widehat{j}) \stackrel{!}{=} T(n - \widehat{j}).$$

c) *(3 pts.)* Using the results from the previous parts show that

$$\begin{aligned} T^\dagger(n) \gamma^0 \gamma^j T(n + \widehat{j}) &\stackrel{!}{=} \eta_j^+(n) \mathbf{1}, \\ T^\dagger(n) \gamma^0 \gamma^j T(n - \widehat{j}) &\stackrel{!}{=} \eta_j^-(n) \mathbf{1}, \\ \eta_j^+(n) = \eta_j^-(n) = \eta_j(n) &= (-1)^{\sum_{i=1}^{j-1} n_i}. \end{aligned}$$

d) *(2 pts.)* Finally show that

$$T^\dagger(n) \gamma^0 T(n) \stackrel{!}{=} (-1)^{\sum_{i=1}^{d} n_i} \gamma^0 = \tilde{\varepsilon}(n) \gamma^0.$$

[3] Ones a discretization is introduced, the color-electric field is the source of the Link operator L introduced in the lecture. A detailed derivation of this can be found in `https://arxiv.org/abs/2105.06019`.

3. Momentum space Dirac operator of Staggered fermions

In this exercise you will perform the Fourier transform of

$$
\begin{aligned}
&D_{\mathrm{Stag}}(r,r',t,t')_{s,s'} \\
&= \delta\left(t-t'\right)\left(\frac{\mathrm{i}}{2a}\sum_{j=1}^{d}\eta_j(s)\left[\delta_{s+\hat{j},s'}(\delta_{r,r'}-\delta_{r-\hat{j},r'})+\delta_{s-\hat{j},s'}(\delta_{r+\hat{j},r'}-\delta_{r,r'})\right]\right. \\
&\qquad\left. +\,\delta_{s,s'}\delta_{r,r'}\left[\mathrm{i}\partial_{t'}-m\tilde{\varepsilon}(s)\right]\right).
\end{aligned}
$$

a) *(4 pts.)* Verify that the fourier transform of $D_{\mathrm{Stag}}(n,m,t,t')_{\alpha,\beta}$ is given by

$$
\begin{aligned}
&D_{\mathrm{Stag}}\left(\vec{p},\vec{q},p^0,q^0\right)_{\alpha,\beta} \\
&= \int \mathrm{d}t \int \mathrm{d}t' \frac{1}{|R|}\sum_{n,m\in R} \mathrm{e}^{-\mathrm{i}\left(p^0 t-\vec{p}\cdot\vec{n}a\right)} D_{\mathrm{Stag}}(n,m,t,t')_{\alpha,\beta}\mathrm{e}^{+\mathrm{i}\left(q^0 t'-\vec{q}\cdot\vec{m}a\right)} \\
&\overset{!}{=} 2\pi\delta\left(p^0-q^0\right)\delta_{\vec{p},\vec{q}}\underbrace{\left[-\sum_{j=1}^{d}\left(\tilde{\Gamma}^j\right)_{\alpha\beta}(\vec{p})\frac{1}{a}\sin\left(p_j\frac{a}{2}\right)-\delta_{\alpha,\beta}p^0-m\left(\tilde{\Gamma}^0\right)_{\alpha,\beta}\right]}_{=\tilde{D}_{\mathrm{Stag}}(p)_{\alpha,\beta}},
\end{aligned}
$$

where we introduced

$$
\begin{aligned}
\left(\tilde{\Gamma}^0\right)_{\alpha\beta} &= \delta_{\alpha,\beta}\tilde{\varepsilon}(\alpha), \\
\left(\tilde{\Gamma}^j\right)_{\alpha\beta}(\vec{p}) &= \mathrm{e}^{\mathrm{i}\frac{a}{2}\vec{p}\cdot(\beta-\alpha)}\eta_j(\alpha)\left[\delta_{\alpha+\hat{j},\beta}+\delta_{\alpha-\hat{j},\beta}\right].
\end{aligned}
$$

b) *(1 pt.)* Show that $\tilde{D}_{\mathrm{Stag}}(p)_{\alpha,\beta}$ can be expressed as

$$
\tilde{D}_{\mathrm{Stag}}(p)_{\alpha,\beta} \overset{!}{=} \left(\Gamma^0\right)_{\alpha\gamma}\left[-\sum_{j=1}^{d}\left(\Gamma^j\right)_{\gamma\beta}(\vec{p})\frac{1}{a}\sin\left(p_j\frac{a}{2}\right)-\left(\Gamma^0\right)_{\gamma,\beta}p^0-m\delta_{\gamma,\beta}\right],
$$

where we defined

$$
\Gamma^0=\tilde{\Gamma}^0 \qquad \Gamma^j(\vec{p})=\tilde{\Gamma}^0\tilde{\Gamma}^j(\vec{p}).
$$

Hint: As shown in the lecture notes $\Gamma^\mu(\vec{p})$ constitute a 2^d dimensional representation of the Clifford algebra when considering equal momenta $\vec{p}$:

$$
\{\Gamma^\mu(\vec{p}),\Gamma^\nu(\vec{p})\}=2g^{\mu,\nu}\mathbf{1}.
$$