Exercises for Lattice Field Theory – Hamiltonian and Lagrangian Methods (physics7520) Sose 2026

L. Funcke, S. Singh, A. Bulgarelli and N. Dichter

1. Grassmann calculus

As discussed in the lecture fermionic degrees of freedom are represented as Grassmann numbers in the path integral formulation of quantum field theory. We here now consider a set of $4N$ Grassmann numbers denoted by

$$\eta_1, \dots, \eta_N, \bar{\eta}_1, \dots, \bar{\eta}_N, \rho_1, \dots, \rho_N, \bar{\rho}_1, \dots, \bar{\rho}_N,$$

which anticommute by definition

$$\{\eta_i, \eta_j\} = \{\eta_i, \bar{\eta}_j\} = \{\eta_i, \rho_j\} = \{\eta_i, \bar{\rho}_j\} = \{\bar{\eta}_i, \bar{\eta}_j\} = \dots = 0 \qquad i, j = 1, \dots, N.$$

Implications of this definition and additional properties relating to differentiation and integration can be found it the lecture notes.

a) *(2 pts.)* Give the Taylor expansion of $f(\bar{\eta}, \eta)$ as well as $e^{a\bar{\eta}\eta}$ where η and $\bar{\eta}$ are Grassmann numbers and $a \in \mathbb{C}$.

b) *(2 pts.)* Show that

$$\int \mathrm{d}\bar{\eta}\mathrm{d}\eta \, e^{-a\bar{\eta}\eta} = a.$$

c) *(6 pts.)* Show that

$$Z = I[A] = \int \prod_{l=1}^{N} (\mathrm{d}\bar{\eta}_l \mathrm{d}\eta_l) \exp\left[-\sum_{i,j=1}^{N} \bar{\eta}_i A_{ij} \eta_j\right] = \det(A).$$

Hint: Use

$$\exp\left[-\sum_{i,j=1}^{N} \bar{\eta}_i A_{ij} \eta_j\right] = \prod_{i=1}^{N} \exp\left[-\bar{\eta}_i \sum_{j=1}^{N} A_{ij} \eta_j\right]$$

and

$$\eta_{j_1} \cdot \dots \cdot \eta_{j_N} = \epsilon_{j_1, \dots, j_N} \eta_1 \cdot \dots \cdot \eta_N,$$

where $\epsilon_{j_1, \dots, j_N}$ is the Levi-Civita symbol.

d) *(4 pts.)* Consider the generating functional

$$Z[\rho,\bar\rho] = \int \prod_{l=1}^{N} (\mathrm{d}\bar\eta_l \mathrm{d}\eta_l)\, \exp\left[-\sum_{i,j=1}^{N} \bar\eta_i A_{ij} \eta_j + \sum_{i=1}^{N} (\bar\eta_i \rho_i + \bar\rho_i \eta_i) \right]$$

and show that

$$Z[\rho,\bar\rho] = \det(A) \exp\left[\sum_{i,j=1}^{N} \bar\rho_i A_{ij}^{-1} \rho_j \right].$$

Hint: For this use the result from part c) and complete the square inside the exponential, similarly to the steps followed Exercise 1.b) on sheet 2.

e) *(2 pts.)* Using the previous results calculate the two point function

$$\langle \eta_n \bar\eta_m \rangle = \frac{1}{Z} \int \prod_{l=1}^{N} (\mathrm{d}\bar\eta_l \mathrm{d}\eta_l)\, \eta_n \bar\eta_m \exp\left[-\sum_{i,j=1}^{N} \bar\eta_i A_{ij} \eta_j \right] = \frac{1}{Z} \frac{\delta^2 Z[J]}{\delta\rho_m \delta\bar\rho_n}\bigg|_{\rho,\bar\rho=0}.$$

f) *(4 pts.)* We will now consider the case of naive free lattice fermions as introduced in the lecture, specifically we consider the Euclidean lattice action

$$S^E[\psi,\bar\psi] = \sum_{n,m} \bar\psi(n) K(n,m) \psi(m),$$

where the matrix elements of K are given by

$$K_{\alpha\beta}(n,m) = \frac{1}{2a} \sum_{\mu} \left(\gamma_\mu^E\right)_{\alpha\beta} \left[\delta_{m,n+\hat\mu} - \delta_{m,n-\hat\mu}\right] + M \delta_{m,n} \delta_{\alpha,\beta}$$

and $\psi(n), \bar\psi(n)$ are now Grassmann numbers and play the role of $\eta_i, \bar\eta_i$. As given in the lecture[1] the two point function is given by

$$\left\langle \psi_\alpha(n) \bar\psi_\beta(m) \right\rangle = K_{\alpha\beta}^{-1}(n,m).$$

The goal in the following is to derive the expression for $K_{\alpha\beta}^{-1}(n,m)$ given in the lecture.

i. Show that in momentum space K can be expressed as

$$K(p,q) = \frac{1}{N} \sum_{n,m} e^{-ipna} K(n,m) e^{iqma} \stackrel{!}{=} \delta_{p,q} \left[i \sum_{\mu} \gamma_\mu^E \frac{1}{a} \sin(p_\mu a) + M \mathbb{1} \right].$$

ii. Verify that

$$K^{-1}(p,q) = \delta_{p,q} \frac{-i \sum_{\mu} \gamma_\mu^E \frac{1}{a} \sin(p_\mu a) + M \mathbb{1}}{\sum_{\nu} \frac{1}{a^2} \sin^2(p_\nu a) + M^2}.$$

[1] and computed by you in the previous part

2. Beta-function and continuum limit

Lattice field theories provide a nonperturbative regularization of quantum field theories. However, at the end of the day, we have to make sure that physical quantities do not depend on the UV-cutoff, in our case the lattice spacing a.
Lattice actions typically depend on a set of bare parameters, such as the coupling constant g or the mass m. These are not physical observables, and indeed they are allowed to change, or "run", with the lattice spacing. As we take the continuum limit $a \to 0$, we have to change $g(a)$ and $m(a)$ accordingly to keep the physics constant.
For simplicity, let us focus on a pure gauge $\mathrm{SU}(N)$ lattice gauge theory. Such theory has a single parameter g. Let $Q(g(a), a)$ be an observable that we can compute in our lattice simulations, that approaches the physical value Q_0 as $a \to 0$

$$\lim_{a \to 0} Q(g(a), a) = Q_0.$$

The requirement that continuum physics is insensitive to the UV-cutoff can be expressed as

$$\frac{\mathrm{d}Q}{\mathrm{d}\log a} = 0 \qquad\qquad \left(\frac{\partial}{\partial \log a} + \frac{\partial g}{\partial \log a}\frac{\partial}{\partial g}\right) Q = 0.$$

The β-function is defined as

$$\beta(g) \equiv -\frac{\partial g}{\partial \log a},$$

and it determines the dependence of the coupling on the lattice spacing. The β-function can be expanded around $g = 0$, and for $\mathrm{SU}(N)$ it reads

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 + \mathcal{O}(g^7),$$

where β_0 and β_1 are positive coefficients $\forall N$.

a) *(2 pts.)* Using the simplification

$$a\frac{\mathrm{d}g}{\mathrm{d}a} = \beta_0 g^3,$$

derive the functions $a(g)$ and $g^2(a)$.

b) *(3 pts.)* Now using the relation

$$a\frac{\mathrm{d}g}{\mathrm{d}a} = \beta_0 g^3 + \beta_1 g^5,$$

show that

$$a\Lambda_L = \exp\left(-\frac{1}{2\beta_0 g^2}\right)\left(\beta_0 g^2\right)^{-\frac{\beta_1}{2\beta_0^2}},$$

where Λ_L is an integration constant. Hint: expand for $g \to 0$ and use the freedom in redefining the integration constant.

c) *(2 pts.)* Invert the previous expression perturbatively to obtain $g^2(a)$. Discuss how to tune g to approach the continuum limit $a \to 0$.

d) *(3 pts.)* The β-function provides a perturbative relation between g and a. In typical setups lattice simulations are not close enough to the continuum limit to be in a perturbative regime. A *nonperturbative scale setting* is then necessary.

Consider the static potential between two quarks at a distance r, $V(r)$. At large values of r, the potential is linear in r

$$V(r) \simeq \sigma r + V_0 + \ldots$$

σ is called the string tension, and it can be measured, for example, at LHC. Its physical value in QCD is around $(440\text{MeV})^2$, and we will assume the same value in pure gauge theory. In lattice simulations, the static quark potential can be extracted from the expectation value of large Wilson loops

$$\langle W(r, L) \rangle \simeq e^{-LV(r)} \qquad \text{for large } L.$$

Design a strategy to set the scale (i.e. determine the relation between a and the bare parameter g) using the physical value of σ.