Exercises for Lattice Field Theory – Hamiltonian and Lagrangian Methods (physics7520)
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1. Generating functional of the free scalar field theory

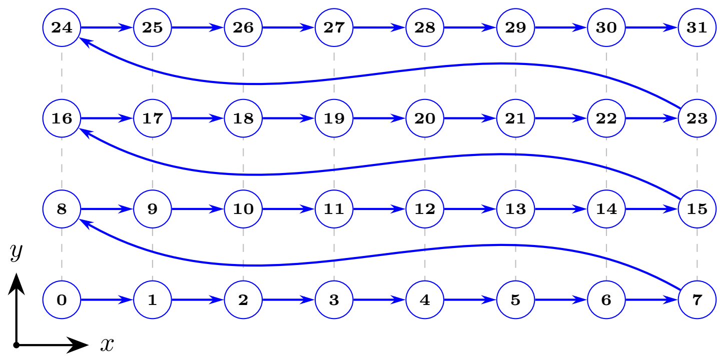
In the lecture you have seen that the Euclidean lattice action of the free scalar field ϕ can be written as

$$S_E = \frac{1}{2} \sum_{n,m} \phi_n K_{nm} \phi_m,$$

where the matrix elements K_{nm} are given by

$$K_{nm} = -\sum_{\mu>0} \left(\delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m} - 2\delta_{n,m}\right) + M^2 \delta_{n,m}.$$

a) *(4 pts.)* The definition of the matrix elements K_{nm} uses a common notation. On the left hand side n (as well as m) denotes an integer and on the right hand side it denotes a vector of integers with the number of components being equal to the space-time dimension. This notation can be best illustrated in the case of a two-dimensional lattice (with periodic boundary conditions) as shown below for the example of an 8×4 lattice.



In this example the left hand side $n = 2$ corresponds to the right hand side $n = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ and the left hand side $n = 21$ corresponds to the right hand side $n = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

i. Show that K is symmetric: $K = K^\top$

ii. Write the 4×4 matrix K for a 2×2 lattice with periodic boundary conditions.

iii. Determine the inverse of this matrix K.
(You can use tools like WolframAlpha, numpy, ... for this.)

b) *(6 pts.)* The lattice partition function is given by

$$Z = \int \prod_l \mathrm{d}\phi_l \exp\left(-\frac{1}{2}\sum_{n,m} \phi_n K_{nm} \phi_m\right),$$

and the generating functional of the free scalar field theory is given by

$$Z[J] = \int \prod_l \mathrm{d}\phi_l \exp\left(-\frac{1}{2}\sum_{n,m} \phi_n K_{nm} \phi_m + \sum_n J_n \phi_n\right).$$

i. Evaluate the gaussian integrals of the generating functional $Z[J]$ to arrive at

$$Z[J] = \left[\det\left(2\pi K^{-1}\right)\right]^{\frac{1}{2}} \exp\left(\frac{1}{2}\sum_{n,m} J_n K^{-1}_{nm} J_m\right).$$

ii. Using the previous result calculate the partition function Z.

iii. Using the previous results calculate the two point function

$$\langle \phi_n \phi_m \rangle = \frac{1}{Z} \int \left(\prod_l \mathrm{d}\phi_l\right) \phi_n \phi_m \exp\left(-\frac{1}{2}\sum_{n,m} \phi_n K_{nm} \phi_m\right) = \frac{1}{Z} \frac{\delta^2 Z[J]}{\delta J_n \delta J_m}\bigg|_{J=0}.$$

2. A covariant derivative on the lattice

In this exercise you will explore why the idea (from continuum gauge theories) of introducing a covariant derivative to ensure gauge invariance can not be carried over naively to a lattice formulation.

a) *(2 pts.)* Consider a scalar field $\phi(x)$ in the continuum which transforms under a U(1) gauge transformation as

$$\phi'(x) = \mathrm{e}^{i\alpha(x)}\phi(x).$$

Show that

$$(D_\mu \phi)'(x) = \mathrm{e}^{i\alpha(x)} D_\mu \phi(x).$$

Where

$$D_\mu = \partial_\mu + iA_\mu$$

is the covariant derivative and the gauge fields A_μ transform under a gauge transformation as

$$A'_\mu(x) = A_\mu - (\partial_\mu \alpha(x)).$$

b) *(6 pts.)* In this part we will consider if the gauge fields A_μ can be introduced after discretizing the derivative $\partial_\mu \phi(x)$. Let us consider the forward difference operator for this discretization

$$\partial_\mu \phi(x) \to \Delta^f_\mu \phi(n) = \frac{\phi(n+\hat{\mu}) - \phi(n)}{a}.$$

Assume the same transformation behavior of the scalar fields ones discretized, namely

$$\phi'(na) = \mathrm{e}^{i\alpha(na)}\phi(na).$$

i. Show that $(\Delta_\mu^f \phi)'(n)$ reproduces the continuum problem

$$(\partial_\mu \phi)'(x) = \partial_\mu \phi'(x) = \mathrm{e}^{i\alpha(x)}(\partial_\mu + i(\partial_\mu \alpha(x)))\phi(x)$$

up to $\mathcal{O}(a)$.
Hint: Taylor expand $\mathrm{e}^{i\alpha((n+\hat{\mu})a)}$ around $x = na$.

ii. Argue why the introduction of a gauge field A_μ is not able to reconcile exact gauge invariance.

3. Wilson gauge action

In the lecture you have seen the Wilson gauge action

$$S_G[U] = \frac{1}{e^2} \sum_{n \in \Lambda} \sum_{\mu < \nu} \mathrm{Re}\,\mathrm{tr}\,[\mathbf{1} - U_{\mu\nu}(n)]\,,$$

where the trace is only necessary for non-abelian gauge theories and the plaquette $U_{\mu\nu}(n)$ is a product of four link variables

$$\begin{aligned} U_{\mu\nu}(n) &= U_\mu(n) U_\nu(n+\hat{\mu}) U_{-\mu}(n+\hat{\mu}+\hat{\nu}) U_{-\nu}(n+\hat{\nu}) \\ &= U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu^\dagger(n+\hat{\nu}) U_\nu^\dagger(n). \end{aligned}$$

3.1 Abelian gauge theory

For an abelian gauge theory (U(1)) the link variables $U_\mu(n)$ are related to the vector potentials $A_\mu(n)$ via

$$U_\mu(n) = \mathrm{e}^{iaeA_\mu(n)}.$$

In this part of the exercise you will show that the Wilson gauge action has the correct naive continuum limit for an *abelian* gauge theory (U(1)).

a) *(1 pts.)* Start off by expressing the plaquette $U_{\mu\nu}(n)$ in terms of the vector potentials. The result should only depend on the discretized field strength tensor

$$F_{\mu\nu}^D(n) = \frac{1}{a}\left(A_\nu(n+\hat{\mu}) - A_\nu(n)\right) - \frac{1}{a}\left(A_\mu(n+\hat{\nu}) - A_\mu(n)\right).$$

b) *(3 pts.)* Insert the result of the previous part into the Wilson gauge action and expand to the first non-vanishing order in a.

c) *(1 pts.)* Take the limit $a \to 0$ and thus show

$$\lim_{a\to 0} S_G[U] = \frac{1}{4} \int \mathrm{d}^4x \sum_{\mu,\nu} F_{\mu\nu} F_{\mu\nu},$$

where $F_{\mu\nu}$ is the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

3.2 Non-abelian gauge theory

For a non-abelian gauge theory (SU(N)) the link variables $U_\mu(n)$ are related to the vector potentials $A^a_\mu(n)$ via

$$U_\mu(n) = \mathrm{e}^{iaeA^a_\mu(n)t^a},$$

where t^a are complex hermitian matrices[1] which satisfy

$$\begin{aligned} \mathrm{tr}\,(t^a) &= 0, \\ \mathrm{tr}\left(t^a t^b\right) &= \delta_{ab}, \\ \left[t^a, t^b\right] &= i f^{abc} t^c. \end{aligned}$$

Here $f^{abc} \in \mathbb{R}$ denote the structure constants of su(N). One often uses the notation

$$A_\mu(n) = A^a_\mu(n) t^a$$

to denote the matrix valued vector potentials.

In this part of the exercise you will show that the Wilson gauge action has the correct naive continuum limit for a *non-abelian* gauge theory (SU(N)).

a) *(4 pts.)* Again start off by expressing the plaquette $U_{\mu\nu}(n)$ in terms of the vector potentials and show that

$$U_{\mu\nu}(n) = \exp\left[ia^2 e F_{\mu\nu}(n) + \mathcal{O}(a^3)\right],$$

where the field strength tensor in the non-abelian case is given by

$$F_{\mu\nu}(n) = \partial_\mu A_\nu(n) - \partial_\nu A_\mu(n) + ie[A_\mu(n), A_\nu(n)].$$

Hint: The Baker-Campbell-Hausdorff formula for the product of exponentials of matrices might be useful in this case

$$\exp(X)\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] + \frac{1}{12}[Y,[Y,X]] + \dots\right),$$

where the dots denote higher order nested commutators. In addition, taylor expanding terms like $A_\mu(n+\hat{\nu}) = A_\mu(x = (n+\hat{\nu})a)$ around $x = na$ will be necessary.

b) *(3 pts.)* Insert the result of the previous part into the Wilson gauge action and expand the exponential up to and including order a^4. Do the terms of order a^2 and a^3 vanish? If so why?
Hint: The generators are traceless and the trace of a commutator of two matrices vanishes:

$$\mathrm{tr}\,([X,Y]) = \mathrm{tr}(XY) - \mathrm{tr}(YX) = \mathrm{tr}(XY) - \mathrm{tr}(XY) = 0.$$

[1] These are the so-called generators of the algebra under consideration (su(N)).