Exercises for Lattice Field Theory – Hamiltonian and Lagrangian Methods (physics7520)
Sose 2026

L. Funcke, S. Singh, A. Bulgarelli and N. Dichter

---

1. Dimensional Analysis

In natural units ($\hbar = c = 1$), every physical quantity can be assigned a *mass dimension* $[Q] = d$ meaning $Q \sim M^d$ for some mass M. We write $[Q]$ for the mass dimension of Q. Since the action $S = \int d^D x\, \mathcal{L}$ must be dimensionless (it appears as e^{-S} in the path integral) and $[d^D x] = -D$ (with D the number of spacetime dimensions), we have

$$[\mathcal{L}] = D.$$

a) *(2 pts.)* Consider the action of a real scalar field ϕ in D Euclidean spacetime dimensions,

$$S = \int d^D x \left[\frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right].$$

  i. Determine $[\phi]$ in terms of D.
  ii. What are the mass dimensions of m and λ?
  iii. For which value of D is λ dimensionless?

b) *(2 pts.)* Consider a Dirac fermion ψ with action

$$S = \int d^4 x\, \bar{\psi}\, (i\partial\!\!\!/ - m)\, \psi.$$

  i. Determine $[\psi]$.
  ii. A Yukawa interaction $g\, \bar{\psi}\phi\psi$ is added. What is the mass dimension of g?

c) *(2 pts.)* Consider a U(1) gauge theory (quantum electrodynamics) in $D = 4$,

$$S = \int d^4 x \left[\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(iD\!\!\!/ - m)\psi \right],$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu = \partial_\mu - ieA_\mu$.

  i. Determine the mass dimension $[A_\mu]$.
  ii. What is the mass dimension of the electric charge e?
  iii. Now consider the same theory in D dimensions. For which D is the gauge coupling e dimensionless?

d) *(2 pts.)* On a D-dimensional hypercubic lattice with spacing a, a scalar field $\phi(x)$ lives on the sites $x = a\,n$ with $n \in \mathbb{Z}^D$. The lattice action is

$$S_{\text{lat}} = a^D \sum_x \left[\frac{1}{2} \sum_\mu \left(\frac{\phi(x+\hat{\mu}) - \phi(x)}{a} \right)^2 + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \right].$$

In order to perform a lattice simulation, all quantities must be expressed in units of the lattice spacing (i.e., as *dimensionless* numbers). Define the dimensionless field $\hat{\phi}$ and the dimensionless couplings $\hat{m}$ and $\hat{\lambda}$ such that

$$S_{\text{lat}} = \sum_x \left[\frac{1}{2} \sum_\mu \left(\hat{\phi}(x+\hat{\mu}) - \hat{\phi}(x) \right)^2 + \frac{\hat{m}^2}{2}\hat{\phi}(x)^2 + \frac{\hat{\lambda}}{4!}\hat{\phi}(x)^4 \right].$$

Express $\hat{\phi}$, $\hat{m}$, and $\hat{\lambda}$ in terms of ϕ, m, λ, and a.

2. Degrees of Freedom and Lattice Geometry

Schematically, a path integral of a lattice field theory can be written as

$$\int \mathcal{D}\phi \; e^{-S(\phi)}, \tag{1}$$

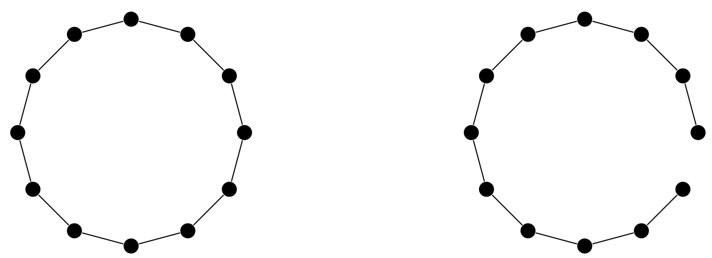
where the integration variable ϕ is a lattice field and S the action defining the theory governing the dynamics of that field. In general the action may couple different fields together. On a discretized space-time lattice, the path integral measure is defined as

$$\mathcal{D}\phi \;\equiv \prod_{i=1}^{N} \mathrm{d}\phi_i, \tag{2}$$

where $\mathrm{d}\phi_i$ defines an integral over the field values ϕ at each of the N lattice sites $i \in \{1, \cdots N\}$.

a) *(2 pts.)* Consider a lattice with N sites and a real scalar field ϕ defined on each site. How many real integration variables does the path integral have?

b) *(2 pts.)* Now consider a Dirac fermion in $D = 4$ dimensions (4 complex components) on N lattice sites. How many real integration variables are present?



c) *(2 pts.)* Consider the previous cartoon, where a one-dimensional lattice with periodic (left) and open (right) boundary conditions are shown. Topologically, a lattice with periodic boundary conditions is a circle, while with open boundary conditions is a segment. In higher dimensions, a lattice with periodic boundary conditions is topologically equivalent to a (hyper-)torus, while with open boundary conditions is a (hyper-)cube.

Given a lattice with N^D sites in D spacetime dimensions, express the number of links in terms of N and D for periodic and open boundary conditions.

3. Discrete Fourier Transform and Lattice Derivatives

Consider a one-dimensional lattice with N sites, lattice spacing a, and periodic boundary conditions, such that lattice momenta are well defined. The lattice sites are $x_n = na$, with $n = 0, \dots, N-1$.

a) *(2 pts.)* Define the discrete Fourier transform by

$$\phi_n = \frac{1}{\sqrt{N}} \sum_p e^{ipna} \tilde{\phi}(p),$$

where the allowed momenta are

$$p = \frac{2\pi}{Na} k, \quad k = 0, \dots, N-1.$$

Show that the inverse transform is

$$\tilde{\phi}(p) = \frac{1}{\sqrt{N}} \sum_n e^{-ipna} \phi_n.$$

b) *(2 pts.)* Consider the forward lattice derivative

$$(\partial\phi)_n = \frac{\phi_{n+1} - \phi_n}{a}.$$

Show that in momentum space this becomes

$$\tilde{\partial}_p = \frac{e^{ipa} - 1}{a}.$$

c) *(2 pts.)* Compute the squared magnitude $|\tilde{\partial}_p|^2$ and show that it can be written as

$$|\tilde{\partial}_p|^2 = \frac{2 - 2\cos(ap)}{a^2},$$

and equivalently as

$$|\tilde{\partial}_p|^2 = \frac{4}{a^2} \sin^2\left(\frac{ap}{2}\right).$$

d) *(2 pts.)* Consider instead the symmetric derivative $(\partial\phi)_n = \frac{\phi_{n+1} - \phi_{n-1}}{2a}$. Show that in momentum space this becomes

$$\tilde{\partial}_p = \frac{i}{a} \sin(ap).$$

e) *(2 pts.)* Expand both expressions for small $ap \ll 1$ and show that they reproduce the continuum derivative.

4. Regularization and UV-finite quantities (2 points)

The purpose of this exercise is to appreciate how a lattice regularization of a path integral allows for regulating divergent quantities in quantum field theory. But this is only a part of the story: eventually, we are interested in removing the lattice, performing a "continuum limit". In order to do that, we have to define appropriate quantities that are well defined in this limit. In this exercise we will discuss a simple example.

Consider a real scalar field with Gaussian action in the continuum

$$S = \frac{1}{2}\int dx\, \phi(x)^2.$$

We define the partition function $Z = \int \mathcal{D}\phi e^{-S}$ and the observables

$$\langle \phi(x)^2 \rangle = \frac{1}{Z}\int \mathcal{D}\phi\, \phi^2(x) e^{-S}, \qquad \langle \phi(x)^4 \rangle = \frac{1}{Z}\int \mathcal{D}\phi\, \phi^4(x) e^{-S}.$$

a) *(2 pts.)* Discretize space into N lattice sites with spacing a:

$$S = \frac{a}{2}\sum_{i=1}^{N} \phi_i^2.$$

Show that the partition function factorizes into independent Gaussian integrals $\int_{-\infty}^{+\infty} \mathrm{d}\phi_i$. Compute the partition function and show that it is divergent both in the continuum ($a \to 0$) and thermodynamic ($N \to \infty$) limit.

b) *(2 pts.)* Compute

$$\langle \phi_j^2 \rangle = \frac{1}{Z}\prod_i \int_{-\infty}^{+\infty} \mathrm{d}\phi_i\ \phi_j^2\ e^{-a\phi_i^2/2}, \qquad \langle \phi_j^4 \rangle = \frac{1}{Z}\prod_i \int_{-\infty}^{+\infty} \mathrm{d}\phi_i\ \phi_j^4\ e^{-a\phi_i^2/2},$$

where j is a specific lattice site. Show that both quantities are divergent in the continuum limit. Hint: it might be useful to remember the identity

$$\int_{-\infty}^{+\infty} \mathrm{d}x\ x^{2m} e^{-x^2} = (-)^m \frac{\partial^m}{\partial \beta^m} \int_{-\infty}^{+\infty} \mathrm{d}x\ e^{-\beta x^2}\bigg|_{\beta=1}$$

c) *(2 pts.)* Define the "Binder cumulant", which is related to the fourth moment of a probability distribution

$$U = \frac{\langle \phi_j^4 \rangle}{\langle \phi_j^2 \rangle^2}.$$

Is U finite in the continuum limit?